Ratios, Proportions, Unit Conversions, and the Factor-Label Method

Math 101, Littlefield[1]

I don't know why, but presentations about ratios and proportions are often confused and fragmented. The one in your textbook is no exception. This handout is an attempt to do better.

First, some definitions. These aren't universal, but they'll be handy for our purposes.

Numbers are things like 6, 0.0039, and 4/11 = 0.3636... Numbers are used to measure and count things. You can do arithmetic with numbers to get more numbers, but they don't actually mean anything until they're placed in some context and associated with some "unit".

Units are things like "feet", "inches", "gallons", "apples", or "days". They tell you what it is that you're measuring or counting.

A **quantity** is what you get when you combine a number and a unit. Examples include "4 feet", "0.0039 inches", "2.5 apples", or "365 days". Quantities are the things we usually care about.

A **ratio** is a comparison between two or more quantities associated with multiplying or dividing (as opposed to adding or subtracting). In English, ratios are often indicated by words like "per" and "for every". Examples include "3.3 workers for every Social Security recipient", "8 slices of pizza for every 3 people", "10 milligrams of drug per 5 cc vial", and "50 miles per hour". Notice that what we usually think of as a *rate* (50 miles per hour) is also a special kind of ratio.

Ratios are only used for relationships where it's reasonable to think that the numbers can be scaled by multiplying or dividing. "3.3 workers for every Social Security recipient"[2] doesn't really mean that somebody cut a worker to pieces — it means that there were 48 million recipients and 158 million workers: 158/48 = 3.3. "50 miles per hour" might mean exactly 50 miles in exactly one hour, but it might also mean 25 miles in half an hour or 200 miles in four hours.

Ratios can be written in a variety of ways, but for most purposes it's easiest to write them as fractions:

$$\frac{3.3\ workers}{recipient} \qquad \frac{8\ slices}{3\ people} \qquad \frac{10\ mg}{5\ cc} \qquad \frac{50\ miles}{hour}$$

A **proportion** is a statement that two ratios are equal. In other words, a proportion specifies that not only is it *reasonable to think* that the numbers in the ratios can be scaled, but that this is actually *true* for the situation at hand.

$$\frac{3.3\ workers}{recipient} = \frac{158 \times 10^6\ workers}{48 \times 10^6\ recipients} \qquad \frac{8\ slices}{3\ people} = \frac{40\ slices}{15\ people} \qquad \frac{10\ mg}{5\ cc} = \frac{7\ mg}{3.5\ cc}$$

Ratios and proportions are valuable because they accurately reflect a huge number of real-life situations. If you know that 3 people eat 8 slices of pizza, and you're planning a party for 15 people, you can use a proportion to figure out how much pizza you need. If you know that a drug comes packaged as 10 mg of

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[2] Social Security numbers for 2005, inferred from http://www.njfac.org/FactsSS.pdf

active ingredient diluted in 5 cc of solution, and you really need a 7 mg dose, you can use a proportion to figure out how many cc to dispense.

Most algebra books treat proportions as something almost trivial: $\frac{a}{b} = \frac{c}{d}$. That equation is easily solved for any of the variables, and you're done. What could possibly be simpler?

What the books overlook is that the difficulty lies in writing a correct proportion to start with.

If you're working the pizza problem and the drug problem, which of these proportions is correct?

$$\frac{8}{3} = \frac{x}{15} \quad or \quad \frac{8}{3} = \frac{15}{x} \;? \qquad\qquad \frac{10}{5} = \frac{y}{7} \quad or \quad \frac{10}{5} = \frac{7}{y} \;?$$

There's certainly no algebraic difficulty solving for *x* and *y* in any of these proportions. *x*, the number of slices of pizza, comes out to be either 40 or about 5.6. And *y*, the amount of drug to dispense, comes out to be either 14 or 3.5.

Clearly one member of each pair must be wrong, but which one?? It is <u>very</u> hard to tell, when the ratios are written with just numbers.

But suppose we include the units when we write the ratios:

$$\frac{8\,slices}{3\,people} = \frac{x\,slices}{15\,people} \quad or \quad \frac{8\,slices}{3\,people} = \frac{15\,people}{x\,slices} \;?$$

$$\frac{10\,mg}{5\,cc} = \frac{y\;\;cc}{7\,mg} \quad or \quad \frac{10\,mg}{5\,cc} = \frac{7\,mg}{y\;\;cc} \;?$$

Well, now it's blindingly obvious — the correct equations are the ones where the units line up: $\frac{slices}{people}$ on both sides of the first, and $\frac{mg}{cc}$ on both sides of the second.

If the units do <u>not</u> line up, then the equations just don't make sense.
$\frac{slices}{people} = \frac{people}{slices}$ or $\frac{mg}{cc} = \frac{cc}{mg}$? I don't think so!

What we've used here is called the "factor-label method" or "dimensional analysis", and it's a life-saver. Without a doubt, it is <u>the</u> most powerful method known for checking that your equations make sense.

So let's formalize it a little bit.

The **factor-label method**, also known as **dimensional analysis**, just means keeping units in the equations and treating them like variables.

That part about "treating them like variables" is important. Just like a <u>variable</u> is a name that stands in for a <u>value</u> you don't happen to know yet, a <u>unit</u> is a name that stands in for a <u>thing</u> you don't really know yet.

(Do you know exactly how long a foot is? Not likely! You have some general idea, and you know that 1 foot = 12 inches by definition, but you don't know exactly what an inch is, either. It's OK to not know exactly what these things are, as long as you keep the names straight when you're working with them.)

As with variables, there are only a few things that you can do with units, and some things that you definitely can **not** do:

$5\ hours + 6\ hours = 11\ hours$	It's OK to add and subtract the same units — just add and subtract the coefficients, and keep the unit the same.
$\frac{90\ miles}{1.5\ \cancel{hours}} \cdot \frac{2\ \cancel{hours}}{1} = 120\ miles$	When the same unit appears on both top and bottom of a fraction, that unit cancels.
$5\ feet\ \cdot 6\ feet = 30\ feet^2$	It's OK to multiply the same units, but you get a new unit. (The area of an rectangle 5 feet by 6 feet is 30 feet^2, more commonly written as "30 square feet".)
$12\ oz\ \cdot 5\ in = 60\ oz\ in$	It's also OK to multiply different units, but both units get carried through into the product. (Shown here is a torque calculation.)
$5\ hours + 12\ feet$ (nonsense!)	It is **not** OK to add and subtract different kinds of units. If you find something like this in the middle of a calculation, you know there's a mistake that needs finding and fixing.
$5\ feet + 4\ inches = 64\ inches$	It's OK to add and subtract quantities that can be converted to some common unit.
$\frac{5\ mg}{cc} = \frac{y\ \ cc}{7\ mg}$ (nonsense!)	It is **not** OK to have different kinds of units on opposite sides of an equation. Here we're looking at weight/volume on the left, but volume/weight on the right. That equation cannot possibly be correct.

More Applications of the Factor-Label Method

As we've seen above, the factor-label method is a very powerful method to help set up correct proportions. It can also be used to help set up other problems involving multiplying and dividing of various quantities. Unit conversion is a classic example.

Let's work a problem involving both a proportion and some unit conversions.

Pesticide Dilution Problem. Instructions for mixing a particular pesticide call for diluting 4 tablespoons of concentrate in 2 gallons of water. But I want only 20 ounces in a spray bottle. How much concentrate do I need to go with 20 ounces of water?

Solution. Problems of this type are always set up as a ratio.

In this case, one good ratio is $\frac{4\ tablespoons\ concentrate}{2\ gallons\ water} = \frac{?\ \ tablespoons\ concentrate}{20\ ounces\ water}$

This ratio seems like it's on the right track because it has $\frac{tablespoons\ concentrate}{(some\ volume\ of)\ water}$ on both sides. But there's a problem with the denominators — on the left side I have "2 gallons" and on the right side I have "20 ounces". The numbers are no problem; they'll just be part of the arithmetic. But I need to convert gallons and ounces to some common unit. Let's go for ounces, since that's where we'd like to end up.

If you Google on "unit conversion", you'll find a bunch of online converters, for example at http://www.onlineconversion.com/ . Those converters will give you the required conversion factor in a single step. For example, http://www.onlineconversion.com will tell you that "1 gallon [US, liquid] = 128 ounce [US, liquid]".

Well, if 1 gallon = 128 ounces, then $\frac{1\ gallon}{128\ ounces} = 1$ and also $\frac{128\ ounces}{1\ gallon} = 1$.

We can of course multiply either side of any equation by 1 without changing the solutions. In this case, it's very helpful to multiply as follows

$$\frac{1\ gallon}{128\ ounces} \cdot \frac{4\ tablespoons\ concentrate}{2\ gallons\ water} = \frac{?\ \ tablespoons\ concentrate}{20\ ounces\ water}$$

Cancelling the units as if they were variables leaves us with

$$\frac{1\ \cancel{gallon}}{128\ ounces} \cdot \frac{4\ tablespoons\ concentrate}{2\ \cancel{gallons}\ water} = \frac{?\ \ tablespoons\ concentrate}{20\ ounces\ water}$$

$$\frac{4\ tablespoons\ concentrate}{128\ \cdot 2\ ounces\ water} = \frac{?\ \ tablespoons\ concentrate}{20\ ounces\ water}$$

Solving the proportion now leaves us with

$$20\ ounces\ water\ \cdot \frac{4\ tablespoons\ concentrate}{128\ \cdot 2\ ounces\ water} = ?\ \ tablespoons\ concentrate$$

$$20\ \cancel{ounces\ water} \cdot \frac{4\ tablespoons\ concentrate}{128\ \cdot 2\ \cancel{ounces\ water}} = ?\ \ tablespoons\ concentrate$$

$$\frac{80}{256} tablespoons\ concentrate = \frac{5}{16}\ tablespoons\ concentrate$$

This is still a little awkward for the real application, since tablespoon measures don't come marked with fractions. Converting again to some smaller unit will help. It turns out that "teaspoon" is a handy smaller unit: $3\ teaspoons = 1\ tablespoon$, or $\frac{3\ teaspoons}{1\ tablespoon} = 1$. As before, we can multiply our answer by this strange form of "1" in order to change the units:

$$\frac{5}{16} \cancel{tablespoons}\ concentrate \cdot \frac{3\ teaspoons}{1\ \cancel{tablespoon}} = \frac{15}{16} teaspoon\ concentrate$$

This number ($^{15}/_{16}$) is close enough to the size of a measuring spoon that I can actually use it to mix the pesticide. So there's our answer: $^{15}/_{16}$ teaspoon of pesticide concentrate in 20 ounces of water.

In more complicated problems, you won't be able to find the required conversion factor in a single step, but you can always use a sequence of steps to get the same effect. For example, converting 30 miles per hour into feet per second:

$$\frac{30\ \cancel{miles}}{\cancel{hour}} \cdot \frac{1\ \cancel{hour}}{3600\ seconds} \cdot \frac{5280\ feet}{\cancel{mile}} = \frac{44\ feet}{second}$$

At this point, you may very well be saying *"Is this man crazy?! There are much easier ways to do unit conversions!"*

If so, you're correct, sort of. For example, we could do the pesticide problem by just substituting equivalents, like this:

$$\frac{4\ tablespoons\ concentrate}{2\ gallons\ water} = \frac{4\ \cdot (3\ teaspoons)\ concentrate}{2 \cdot (128\ ounces) water} = \frac{12\ teaspoons\ concentrate}{256\ ounces\ water}$$

Now we can set up and solve the proportion.

$$\frac{12\ teaspoons\ concentrate}{256\ ounces\ water} = \frac{?\ teaspoons\ concentrate}{20\ ounces\ water}$$

$$20\ ounces\ water\ \cdot\ \frac{12\ teaspoons\ concentrate}{256\ ounces\ water} = \frac{240}{256} teaspoons\ concentrate$$

Finally, reducing the fraction $\frac{240}{256}$ to lowest terms gives us the same answer we had before, $^{15}/_{16}$ teaspoon of pesticide concentrate in 20 ounces of water.

Is this simpler?

Well, the arithmetic ends up being exactly the same. (Of course it would have to be!)

Substituting equivalents does require less writing, perhaps quite a lot less if you're willing to erase and overwrite:

$$\frac{4\ \overset{(3\ teaspoons)}{\cancel{tablespoons}}\ concentrate}{2\ \underset{(128\ ounces)}{\cancel{gallons}}\ water} = \frac{12\ teaspoons\ concentrate}{256\ ounces\ water}$$

In exchange, the factor-label method — "multiplying by strange forms of 1"— provides a **framework** that is able to handle a wide variety of other problems far beyond simple unit conversion.

To apply the factor-label method, we first build a table of quantities that correspond to each other. Eventually, we rewrite each row of the table as a fraction that is a "strange form of 1", and multiply them together. This whole process is driven by looking at the units. What do we have when we start? What do we need when we end? From the starting units, we pick correspondences that cancel units (factor labels) we don't want, while retaining or producing units we do want.

Consider, for example, this homework problem:

> 100 Percent of Daily Allowance of Iron (http://mathforum.org/library/drmath/view/58011.html)
> A common foodstuff is found to contain .00125% iron. The serving size is 87.0 grams. If the recommended daily allowance is 18mg of iron, how many servings would a person have to eat to get 100% of the daily allowance of iron?

What we are essentially asked to do is a complicated unit conversion that turns "100% daily allowance of iron" into "number of servings".

So, we want to set up a sequence of multiplications by "strange forms of 1" that looks like this:

$$\frac{100\%\ daily\ allowance\ of\ iron}{1} \times \text{strange 1} \times \text{strange 1} \times \text{strange 1} \times \ldots = \frac{??\ servings}{1}$$

First, we read the problem and build a table of things that correspond to each other:

Quantity #1	Quantity #2	Why?
.00125 gm iron	100 gm foodstuff	Given, and definition of percent as parts per 100. I'm using "gm" because that's the unit used elsewhere in the problem, and I'm assuming that the given percentage is "by weight".
1 serving	87.0 gm foodstuff	Given
daily allowance iron	18 mg iron	Given

OK, right off we're in trouble because there's nothing in the table that corresponds to that leading "100%". No problem, we can handle that by just converting it decimal form as usual. That gets us to this:

$$\frac{1.00\ daily\ allowance\ iron}{1} \times \text{strange 1} \times \text{strange 1} \times \text{strange 1} \times \ldots = \frac{??\ servings}{1}$$

Now we have "daily allowance iron" on top, so we need something to cancel that out. Looking through the table, there's only one correspondence that applies. So we write that one as a fraction and stick it in as a "strange 1":

$$\frac{1.00\ daily\ allowance\ iron}{1} \times \frac{18\ mg\ iron}{daily\ allowance\ iron} \times \text{strange 1} \times \text{strange 1} \times \ldots = \frac{??\ servings}{1}$$

The unmatched unit is now "mg of iron". Looking through the table, we don't have anything involving that exact quantity, but we do have "gm of iron". From general knowledge, we know how to convert mg to gm, and then we can use the correspondence we know between gm of iron and gm of foodstuff:

$$\frac{1.00\ daily\ allowance\ of\ iron}{1} \times \frac{18\ mg\ of\ iron}{daily\ allowance\ of\ iron} \times \frac{0.001\ gm}{1\ mg} \times \frac{100\ gm\ foodstuff}{.00125\ gm\ iron} \times \ldots = \frac{??\ servings}{1}$$

The unmatched unit is now "gm foodstuff", and we have a correspondence to handle that.

$$\frac{1.00\ daily\ allowance\ of\ iron}{1} \times \frac{18\ mg\ of\ iron}{daily\ allowance\ of\ iron} \times \frac{0.001\ gm}{1\ mg} \times \frac{100\ gm\ foodstuff}{.00125\ gm\ iron} \times \frac{1\ serving}{87.0\ gm\ foodstuff} = \frac{??\ servings}{1}$$

I think we're done now. Let's see...

$$\frac{1.00\ \cancel{daily\ allowance\ of\ iron}}{1} \times \frac{18\ \cancel{mg\ of\ iron}}{\cancel{daily\ allowance\ of\ iron}} \times \frac{0.001\ \cancel{gm}}{1\ \cancel{mg}} \times \frac{100\ \cancel{gm\ foodstuff}}{.00125\ \cancel{gm\ iron}} \times \frac{1\ serving}{87.0\ \cancel{gm\ foodstuff}} = \frac{??\ servings}{1}$$

Doing the arithmetic gives us our answer: 16.6 servings

$$\frac{1.00}{1} \times \frac{18}{1} \times \frac{0.001}{1} \times \frac{100}{.00125} \times \frac{1\ serving}{87.0} = \frac{16.6\ servings}{1}$$

Without the labels, that computation looks like black magic. With the labels, it's pretty straightforward.

We can check the answer by working backwards. 16.6 servings would be 16.6*87 = 1444 gm of foodstuff. At 0.00125% iron, 1444 gm foodstuff * 0.00125/100 gives 0.018 gm iron = 18 mg of iron, and that's quoted as being the daily allowance. Great, that checks!

Let's try another one:

> Administering Insulin http://mathforum.org/library/drmath/view/63328.html
> If a doctor prescribes 30 units of insulin in 500 ml to be administered over 2 hours, how many drops per minute should be administered if the set is calibrated to deliver 20 drops per ml?

Our answer is supposed to be "drops per minute". What have we got to work with?

First, we need to think about the physical situation. The doctor has asked for a bag of insulin solution to be infused into a patient. The bag contains 30 units of actual insulin, dissolved in 500 ml. The entire 500 ml is supposed to be delivered, over a period of 2 hours. Thus, what we're really given is one rate, 500 ml per 2 hours, and asked to turn it into so many drops per minute.

The fact that the bag contains 30 units of insulin is unnecessary information. You need to get used to having more information than you need — in real-life problems that will almost always be the case. Only in story problems has the author already whittled down the problem to its essentials, or close to them.

OK, now we know where we need to start and where we need to end up:

$$\frac{500\,ml}{2\,hours} \times \text{strange 1} \times \text{strange 1} \times \ldots = \frac{??\,drops}{minute}$$

Let's see what else we have to work with. Let's work the problem in our heads, while building a table of correspondences.

Quantity #1	Quantity #2	Why?
1 ml	20 drops	given as calibration for the set, use this to cancel the "ml"
1 hour	60 minutes	standard unit conversion, use this to cancel the "hours"

Well, gee, that was quick!

$$\frac{500\,ml}{2\,hours} \times \frac{20\,drops}{1\,ml} \times \frac{1\,hour}{60\,minutes} = \frac{83.3\,drops}{minute}$$

Of course, there's a patient at stake on this one. We'll definitely want to check that the answer makes sense! 83.3 drops/minute means 83.3/20 ml per minute, just a hair over 4 ml/minute. 2 hours is 2*60 = 120 minutes. 120 minutes * 4 ml/minute would be 480 ml, and we're a hair over that. OK, sounds good.

It's important to notice that each "strange 1" is really capturing the concept of "corresponds to", which may be quite a different relationship from "is equal to".

When we write $\frac{8\ slices\ pizza}{3\ people}$ and act as if $\frac{8\ slices\ pizza}{3\ people} = 1$, we do not mean that you can substitute 8 slices of pizza for 3 people in any situation whatsoever.

Instead, the notation just means that for the purposes of this problem, if you multiply by that fraction you'll get another true statement.

There are some important assumptions hidden behind that statement.

Basically, it assumes that the fraction is a ratio, so the numbers can be scaled. That's not always true. For example 25°C = 77°F, but it's not valid to use $\frac{25℃}{77℉} = 1$ in the factor-label method because if you scale the numbers, you get false statements. 50°C = 122°F, not 154°F as simple scaling would predict.

In real-life problems, some of the correspondences will come from simple unit conversions. In those, the relationship really is "equals to". 12 inches = 1 foot because each side describes the very same length.

Other correspondences will come from relationships that are only valid in the context of the problem. We already spoke of pizzas and people, mg per daily allowance, and gm per serving. Other common examples are gallons per tank, watts per lightbulb, dollars per kilowatt-hour, and so on.

By the way, one last piece of notation... If you see a unit like "dollars per person per week", the appropriate fractional unit looks like this:

$$\frac{dollars}{person\ week}$$

If you're good at PEMDAS, you can remember this from the standard order of operations:
dollars / person / week = (dollars / person) / week.

If you're not good at PEMDAS, you can figure it out by asking "What makes sense when I try to use this?"

5 persons at 6 dollars per person per week would be 30 dollars per week, and 7 weeks at 6 dollars per person per week would be 42 dollars per person.

Both "person" and "week" have to be in the denominator for things to work out.
